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Quantum decay and the Mandelstam–Tamm time–energy inequality

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Abstract. The Mandelstam–Tamm time–energy inequality is exploited to obtain a transparent expression of the lifetime–energy uncertainty relation for decaying quantum systems along with some useful features of the quantum non-decay probability.

It is frequently stated that, for a metastable quantum state, the time–energy uncertainty relation (TEUR) is manifested in the form of an *equality* connecting the lifetime and 'width' (Landau and Lifshitz 1977, Bauer and Mello 1978). Basically, such an assertion follows from the well known treatment of decay due to Weisskopf and Wigner (1930) where the width at half-maximum of the Lorentzian line shape was related to the average life of the associated exponential decay. But, since the quantum non-decay probability can *never* be purely exponential in nature (Ersak 1969, Fonda *et al* 1978), unless there exists some kind of *openness*, it seems desirable to have an explicit form of the TEUR involving the lifetime for a closed, decaying quantum system. Recently, Bauer and Mello (1978) analysed the problem in detail and concluded that neither the formulation of the TEUR due to Mandelstam and Tamm (1945) nor the one proposed by Wigner (1972) is suitable to arrive at the lifetime–width relation for a decaying state; rather, they opined, to achieve this end, one has to talk of 'spreads' in terms of 'equivalent widths' (Bauer and Mello 1976). However, it is difficult to understand *why* the width of the line shape should be so fundamental a quantity to appear in the TEUR, keeping aside the rather unfamiliar measure of spreads involved. So, we feel obliged to reinvestigate the case. To proceed, we shall follow the Mandelstam–Tamm (MT) scheme.

The primary difficulty with the MT inequality (Messiah 1976)

$$T_{A_i} = \Delta A_i / |\mathrm{d}\langle A_i \rangle / \mathrm{d}t| \geq \hbar / 2\Delta E \qquad (1)$$

is that the time T_{A_i}, though it is said to refer to some *time* characteristic of the observable corresponding to the operator A_i, does not correspond to the *physical time* t of evolution of the system with respect to some arbitrarily chosen time of its preparation in some particular 'packet' state; it possesses merely the dimension of time. Hence, although we can, in principle, find out the shortest time T_{A_m} from among the set of observables $\{A_i\}$ for the system, T_{A_m} cannot probably 'be considered as a characteristic time of evolution of the system itself' (Messiah 1976). It is also understandable now that the MT time would *not* correspond to the lifetime for a decaying state; an explicit demonstrative calculation led Bauer and Mello (1978) to the same conclusion.

We know that a quantum system, prepared at $t=0$ in some non-stationary state $|\phi\rangle$, evolves causally according to the equation

$$|\phi_t\rangle = e^{-iHt/\hbar}|\phi\rangle, \qquad \phi \in D(H), \tag{2}$$

where $H, \neq H(t)$, is the Hamiltonian of the system. If $|\phi\rangle$ is expressible as an integral over the continuous energy-eigenstates of H the system would decay (see e.g. Davydov 1976) and then the quantity

$$P_t = |\langle\phi|\phi_t\rangle|^2 \tag{3}$$

would define the so-called quantum non-decay probability. It is easily seen from (2) and (3) that P_t has to be an *even function of time*, a property which alone excludes immediately not only the possibility of an exponential decay (see also Fonda *et al* 1978) but also other monotone decreasing *odd* functions from being P_t. Now, choosing A_i in the form of a projector

$$A_i = |\phi\rangle\langle\phi|, \tag{4}$$

we find from (1) the inequality

$$[P_t(1-P_t)]^{1/2}/|dP_t/dt| \geq \hbar/2\Delta E. \tag{5}$$

Remembering that ΔE has to be finite (for $\|H\phi\|$ is finite, $\phi \in D(H)$), the above inequality shows several interesting features to be discussed in what follows.

Firstly, rearranging (5), we obtain

$$|dP_t/dt| \leq (2\Delta E/\hbar)[P_t(1-P_t)]^{1/2} \tag{6}$$

which marks the time T_h when $P_t = \frac{1}{2}$, i.e. the half-life, as a significant time, for then only the right-hand side attains its maximum value so that

$$|dP_t/dt| \leq \Delta E/\hbar \tag{7}$$

always holds; the equality in (7), however, may hold only at T_h. The message of (7) is that *no unstable quantum system can decay completely within a time* $\hbar/\Delta E$. This time, though it is a rather crude estimate (see below), clearly establishes a *limit* to the instability of a decaying quantum system in the time sense.

Secondly, from (6), we obtain some characteristics of quantum decay for different regions of time (note that as $t \to 0$, $P_t \to 1$ and as $t \to \infty$, $P_t \to 0$):

$$|dP_t/dt| = 0, \qquad t = 0, \tag{8a}$$

$$d\cos^{-1} P_t/dt \leq 2\Delta E/\hbar, \qquad t \to 0, \tag{8b}$$

$$dP_t^{1/2}/dt \leq \Delta E/\hbar, \qquad t \to \infty, \tag{8c}$$

and, in general,

$$d\cos^{-1} P_t^{1/2}/dt \leq \Delta E/\hbar, \qquad 0 \leq t \leq \infty. \tag{8d}$$

These results are likely to be of interest in small- and large-time studies of the behaviour of P_t.

Thirdly, integrating (8*d*) directly, one obtains

$$t \geq (\hbar/\Delta E)\cos^{-1} P_t^{1/2} \tag{9}$$

so that the *minimum-time limit*, stated earlier, turns out to be *precisely* $\pi\hbar/2\Delta E$. Also, (9) leads straightforwardly to the inequality

$$P_t \geq \cos^2(\Delta E t/\hbar), \qquad 0 \leq t \leq \pi\hbar/2\Delta E, \tag{10}$$

obtained also by Fleming (1973) through a different and lengthy route. From (10) we find, noting $P_{T_h} = \frac{1}{2}$, that

$$\Delta E T_h \geq \pi\hbar/4, \qquad 0 < T_h \leq \pi\hbar/2\Delta E. \tag{11}$$

On the other hand, if $T_h > \pi\hbar/2\Delta E$, we have

$$P_t \geq 0 > 1 - 2\Delta E t/\pi\hbar, \qquad t > \pi\hbar/2\Delta E, \tag{12}$$

from which it again follows that

$$\Delta E T_h > \pi\hbar/4, \qquad T_h > \pi\hbar/2\Delta E. \tag{13}$$

Joining (11) and (13), we obtain the desired TEUR for a decaying quantum system:

$$\Delta E T_h \geq \pi\hbar/4. \tag{14}$$

This is an *inequality* and clearly shows that a *physical* time, the half-life, and the 'root mean square' deviation of energy *are related*. The fact that the 'width' is not a very dependable quantity as a measure of the energy spread, despite the *experimental* significance it bears, is understandable from the observation that for a Lorentzian line shape, though the width is finite, ΔE turns out to be infinite, pointing to the ill defined nature of the prepared state ($\phi \notin D(H)$).

Fourthly, we can use (6) to obtain also a *lower bound* to P_t over a time interval not contained in (10) as follows. For $t \leq T_h$, $\frac{1}{2} \leq P_t \leq 1$, so that the inequality

$$|\mathrm{d}P_t/\mathrm{d}t| \leq 2\Delta E P_t/\hbar \tag{15}$$

holds and this, on integration gives

$$P_t \geq \mathrm{e}^{-2\Delta E t/\hbar}, \qquad 0 \leq t \leq T_h. \tag{16}$$

Thus, for unstable packets with not-too-small lifetimes (i.e. $T_h > \pi\hbar/2\Delta E$), the inequality (16) would be useful in studying the small-time behaviour of P_t (for a discussion, see Ghirardi *et al* (1979)) in the range $\pi\hbar/2\Delta E \leq t \leq T_h$.

Fifthly, focusing our attention on the 'intermediate times' over which P_t is usually said to follow an exponential character (Chiu *et al* 1977, Fonda *et al* 1978), we denote by T the time which corresponds to the extremum of the function $-\ln P_t/t$ and find that

$$T = (P_T \ln P_T)/\mathrm{d}P_t/\mathrm{d}t|_T \tag{17}$$

which, by virtue of (6), leads to the inequality

$$T \geq (\hbar/2\Delta E)[-\ln P_T(1/P_T - 1)^{-1/2}]. \tag{18}$$

Understandably, it is only over a region *around* T where P_t would behave in an almost-exponential fashion. Noting that the square bracketed part of (18) is bounded from above and that for small enough times $-\ln P_t/t \approx \Delta E^2 t/\hbar^2$, coupled with (7), one can now make the following remark. If ΔE is large, T may be sufficiently small and the region over which P_t behaves near-exponentially may be small, but, for small ΔE, T would be considerably larger, as would the desired near-exponential region. From a detailed analysis, emphasising particularly the *preparation* of the unstable state, Fonda *et al* (1978) arrived at the same conclusion.

As an example, let us study the evolution of a one-dimensional Gaussian packet in field-free space. Our calculation shows that for the wavefunction

$$\phi(x) = \sigma^{-1/2}\pi^{-1/4} \exp(-x^2/2\sigma^2) \tag{19}$$

the corresponding energy form-factor $\omega(E)$ is given by

$$\omega(E) = (2m/\pi)^{1/2}\sigma\hbar^{-1}E^{-1/2} \exp(-2m\sigma^2 E/\hbar^2) \tag{20}$$

and the quantum non-decay probability P_t turns out to be

$$P_t = \left|\int \omega(E)\, e^{-iEt/\hbar}\, dE\right|^2 = (1 + 2\Delta E^2 t^2 \hbar^{-2})^{-1/2} \tag{21}$$

where

$$\Delta E = \hbar^2/8^{1/2}m\sigma^2. \tag{22}$$

From (21), we obtain immediately

$$\Delta E T_h = (\tfrac{3}{2})^{1/2}\hbar \tag{23}$$

as a manifestation of the TEUR. It may also be seen from (21) that for large times P_t varies as t^{-1} and not as t^{-3} which was expected to be a more-or-less general behaviour from the work of Chiu *et al* (1977). For such a packet we have found that the near-exponential behaviour prevails around $T \approx 2^{1/2}\hbar/\Delta E$ and hence the variation of T with ΔE is in accordance with our remark towards the end of the preceding paragraph.

Finally, we wish to mention that Fleming (1973) also obtained an uncertainty-type relation using (10) and involving the so-called 'average life' of the unstable state. But his result is of limited validity, for there can be many unstable quantum states which do not possess a *finite* average life when defined in the way he had chosen (see also other references in Fleming (1973)); the example given above is just one of them. Once again, this reflects the fact that our classical notions may not always be suitable for proper quantum descriptions.

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